

# Free fall onto evaporating black holes at the quantum limit

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Black hole space times evaporate in discrete steps due to remarkably slow Hawking radiation. We here identify evaporation with essentially extremal states at the limit of quantum computation, performing  $2.7 \times 10^{79}$  bit calculations per photon emission in a one solar mass black hole. During evaporation, particles in free fall co-evolve satisfying  $EM = \text{constant}$ , where  $E$  and  $M$  denote the total mass energy-at-infinity of the particle and, respectively, black hole. Particles are hereby increasingly entangled with the black hole space-time over the course of its evaporation.

## I. INTRODUCTION

The evaporation of black holes is a singular perturbation away from thermal equilibrium described by power laws in luminosity  $L$  and temperature  $T$  as a function of total mass energy-at-infinity  $Mc^2$ , where  $c$  denotes the velocity of light, i.e.,

$$L \propto M^{-p}, \quad T \propto M^{-q} \quad (1)$$

with  $p, q > 0$  that is characteristic for open self-gravitating systems with negative specific heat. Evaporation by Hawking radiation [3] shows  $p = 2$  and  $q = 1$ . Classical systems such as globular clusters are qualitatively similar with  $p \simeq 0.9$  and  $q \simeq 6$  [14]. For Hawking radiation [3], emission of individual photons is once every few thousand light crossing time scales [1, 5, and below]. A Schwarzschild given by the line-element

$$ds^2 = -\alpha^2 dt^2 + \frac{dr^2}{\alpha^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (2)$$

hereby changes in essentially discrete steps by individual photon emissions one-by-one

$$\dot{M}(t) = - \sum_{i=1}^N \epsilon_i \delta(t - t_i) \quad (3)$$

at instances  $t_i$ , where  $\epsilon_i$  denotes the energy of the photon emitted measured at infinity,  $\delta(t)$  is the Kronecker delta function,  $\alpha = \sqrt{1 - 2R_g/r}$  is the redshift factor,  $R_g = 2GM/rc^2$  denotes the gravitational radius,  $G$  is Newton's constant and  $c$  is the velocity of light.

As solutions to classical general relativity, black holes hide astronomical amounts of information given by the Bekenstein-Hawking entropy [2-4]

$$k_B^{-1} S = \frac{1}{4} A_H l_p^{-2}, \quad A_H = 4 \ln 3 n l_p^2 \quad (4)$$

in  $n$  Planck sized surface elements  $l_p^2$  for a black hole surface area  $A_H = 16\pi R_g^2$ , where  $l_p = \sqrt{G\hbar/c^3}$  denotes the Planck length in terms of Planck's constant  $\hbar$  and the Boltzmann constant  $k_B$ .

In the  $S$ -matrix formalism (e.g. [9]), the complete evaporation of the black hole represents a unitary evolution. It implies that all of  $S$  in (4) is ultimately recovered in information projected onto the celestial sphere

by Hawking radiation, at times following the evaporative lifetime  $t_{ev}$  of the black hole. In the continuum approximation, the black hole evolution satisfies the finite-time singularity

$$M(t) = M_0 \left( 1 - \frac{t}{t_{ev}} \right)^{\frac{1}{3}} \quad (5)$$

for an initial mass  $M_0$  at the Hawking evaporation time

$$t_{ev} = 1280\pi t_c k_B^{-1} S, \quad (6)$$

where  $t_c = R_g/c$  denotes the light crossing time scale of the black hole. In (6),  $t_{ev}$  scales with  $k_B^{-1} S$ , giving the well-known astronomically large time scales for macroscopic black holes. (For globular clusters,  $t_{ev}$  scales with the relaxation time, giving typical scales on the scale of the Hubble time.)

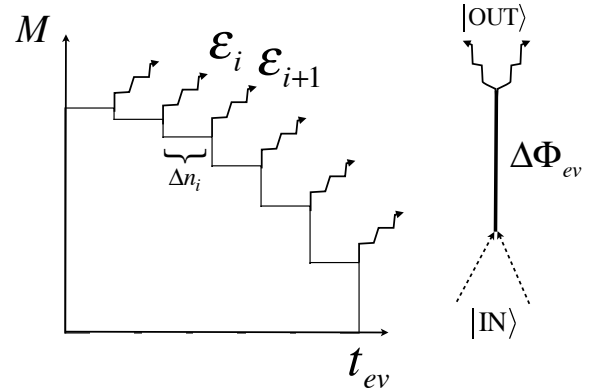


FIG. 1: Schematic overview of the discrete evolution of a Schwarzschild black hole space-time of mass  $M$  from sequences of computations  $\Delta n_i$  each leading to a photon emission. The cumulative result is a finite phase change  $\Delta\Phi_{ev} \simeq Mc^2 t_{ev}/\hbar$  over its evaporative lifetime  $t_{ev}$ .

Here, we shall identify the exceedingly slow and discrete evaporation process with a strict limit of quantum computation in unitary evaporation in the  $S$ -matrix formalism  $|\text{OUT}\rangle = \hat{S} |\text{IN}\rangle$  (Fig. 1), wherein

$$\dim \hat{S} = (k_B^{-1} S) \times (k_B^{-1} S) \quad (7)$$

represents the total number of calculations over the course of complete evaporation, producing state in remnant Hawking radiation recovering the information (4). In this process, the rate of computations is bounded by 4 times the angular velocity of the wave function of the black hole space time, defined by its mass  $M(t)$  [7]

$$\dot{n} = \frac{2Mc^2}{\pi\hbar} = 1.09 \times 10^{81} \left( \frac{M}{M_\odot} \right) \text{ bit s}^{-1} \quad (8)$$

with an associated total computation of

$$\frac{2}{\pi} \Delta\Phi_{ev} \simeq t_{ev} \dot{n} \text{ bits.} \quad (9)$$

Thus, *unitarity prohibits arbitrarily fast evaporation*.

To probe of the evolution of the black hole during evaporation, we consider a perturbation by dropping in a particle of mass  $E$  (posterior to the black hole formation). While an observer in free fall onto a black hole is not aware of an event horizon nor, probably, any associated Hawking radiation, it is aware of a time-dependent tidal field. Defined by the square root of the Kretschmann invariant [6]

$$Q = 4\sqrt{3} \frac{R_g}{r^3}, \quad (10)$$

$Q$  is a scalar field that may invariably be parameterized by the observers' eigentime  $\tau$  or time-at-infinity  $t$ . An observer in free fall is hereby aware of  $M$  upon measuring  $Q$ . Observers  $\mathcal{O}'$  at fixed Schwarzschild radius with zero angular momentum (ZAMO's [13]) notice  $M$  evolving with respect to their eigentime  $\tau$  according to  $(dM/d\tau)_{ZAMO} = -L/\alpha$ . Observers  $\mathcal{O}$  in free fall detect  $dM/d\tau$  be greater than the ZAMO rate by an additional Lorentz factor. To be sure,  $\mathcal{O}$  hereby concludes that  $M$  is rapidly evolving, not constant as is customary assumed (see also [16] for a discussion). To evaluate the evolution of the Schwarzschild space-time observed in free fall, we set out to solve for the geodesic trajectories, taking into account jump conditions associated with the discrete evolution sketched in Fig. 1. We hereby shall identify  $M(\tau)$  and a correlation with  $\mathcal{O}$ 's total energy at infinity  $E(\tau)$ .

In §2, we identify  $t_{ev}$  with extremal quantum computation in unitary evaporation. During evaporation, we solve for jump conditions in geodesic motion of massive observers subject to (3) in §3, that are summarized by a new conserved quantity that expresses a co-evolution of  $E$  and  $M$  (§4). In §5, we give some concluding remarks on entanglement of  $\mathcal{O}$  with the black hole space time.

## II. EXTREMAL QUANTUM COMPUTATION

The energy spectrum of Hawking radiation photons is effectively that of a black body at the finite horizon temperature

$$k_B T_H = \frac{\hbar c}{8\pi R_g}, \quad (11)$$

giving an average photon energy  $\epsilon_\gamma = C k_B T_H$ . For black body radiation, Planck's law  $u = (2\hbar c^{-2} \nu^3 / (e^x - 1))$ ,  $x = \hbar\nu/k_B T$ , for the energy density in radiation per unit solid angle and frequency  $\nu$  implies

$$C = \frac{f(3)}{f(2)}, \quad f(p) = \sum_0^\infty \frac{x_n^p}{e^{x_n} - 1} \Delta x, \quad (12)$$

where  $x_n = n\Delta x$  and  $\Delta x = \hbar\Delta\nu/k_B T$  denotes a spacing in dimensionless frequency. In the continuum limit obtain  $C = 2.7012$ . However, the temperature (11) represents an energy level spacing defined by the horizon quantization (4). With  $\hbar\Delta\nu = k_B T \log 3$ ,  $C = 2.7955$ . The photon emission rate  $\dot{N} = L/\epsilon_\gamma$  associated with the luminosity  $L = \hbar c^2 / 15360\pi R_g^2$  of black body radiation at (11), satisfies

$$\dot{N}_\gamma = \frac{1}{1920C} t_c^{-1} \simeq \frac{1}{5300} t_c^{-1}, \quad (13)$$

where  $t_c = R_g/c$  denotes the light crossing time. This emission rate of one photon per few thousand light crossing time scales is a key property of Hawking radiation, previously identified in [1]. This emission frequency satisfies

$$\nu \simeq 40 \left( \frac{M}{M_\odot} \right)^{-1} \text{ Hz.} \quad (14)$$

The estimates (13-14) show a discrete evolution (3) in steps of size  $\epsilon = C k_B T$ .

If each photon carries of  $k$  bits to infinity, the luminosity  $L = -\dot{M}c^2$ , the total amount of information received is

$$I = k \log 2 \int_0^{t_{ev}} \dot{N} dt = \frac{k \log 2}{C} \int_0^M \frac{c^2 dM}{\epsilon_\gamma} \simeq \frac{k}{4} k_B^{-1} S(15)$$

within 1% in the above mentioned approximation  $C = 2.7955$ . Our information extraction scheme hereby follows the power law

$$I(t) \simeq \frac{k}{4} k_B^{-1} S \left( \frac{t}{t_{ev}} \right)^{\frac{2}{3}} \quad (0 \leq t \leq t_{ev}) \quad (16)$$

to within the same precision.

During evaporation, the black hole is susceptible to small kicks received in the process of emitting each photon. As a result, the net momentum the black hole obtains from a random walk in momentum space. Let  $\delta\alpha = \epsilon_\gamma/Mc^2$  denote the angular perturbation introduced by emitting one photon. Over the course of complete evaporation, it grows by a random walk to about  $\Delta\alpha = \sqrt{N} \delta\alpha$ , i.e.,

$$\Delta\alpha \simeq \frac{A_n}{l_p^2} \times \frac{\hbar c}{8\pi R_g M c^2} \simeq 2. \quad (17)$$

A  $k = 4$  encoding [cf. 15]. comprising polarization and direction (in photon momenta) would hereby be effective over the lifetime of the black hole, thus allowing in

principle all information in (4) to be projected onto the celestial sphere.

In unitary evaporation with information recovery according to (15),  $t_{ev}$  is bounded below by (7-8),

$$t_{ev} \simeq \frac{\dim \hat{S}}{\dot{n}} = 2\pi t_c k_B^{-1} S. \quad (18)$$

This bound explains the long duration (6). Specifically, the number of calculations involved in preparing for a photon emission satisfies

$$\Delta n = \frac{\dot{n}}{\dot{N}_\gamma} = \frac{960C}{\pi^2} k_B^{-1} S = 2.7 \times 10^{79} \left( \frac{M}{M_\odot} \right) \text{ bits.} \quad (19)$$

In the above, we have neglected information in any accompanying Hawking radiation in gravitational waves, as its luminosity is mere 1% of that in electromagnetic radiation [8]. (Also, gravitational waves are notoriously difficult to detect.) Conceivably, small perturbations such as by these to the evolution of the black hole may account for our estimate  $\alpha$  in (15) being slightly less than one by a similar amount. Furthermore, as pointed out in [4], surface discretization of the horizon area points to Shannon entropy in units of  $\log 3 l_p^2$ , where  $\log 3$  refers to three states in each Planck sized unit. The relation of the third mode, beyond the two in the present 4 bit-encoding, remains elusive, perhaps highlighting our lack of understanding of the full quantum mechanical Hamiltonian describing the evolution of the black hole space-time during its evaporation. Nevertheless, the overall agreement within about one percent between information extraction by Hawking radiation and entropy in the event horizon suggests that electromagnetic radiation alone essentially accounts for unitary evolution.

### III. JUMP CONDITIONS IN FREE FALL

In what follows, we use geometrical units ( $G = c = 1$ ), whereby  $R_g = M$ . We assume that the observer  $\mathcal{O}$  has a total energy satisfying the accretion condition  $k_B T < E \ll M$ . In the Schwarzschild line-element,  $\mathcal{O}$ 's velocity along a radial trajectory satisfies

$$u^b = (\dot{t}, \dot{r}, 0, 0) = (\alpha^{-1} \cosh \mu, -\alpha \sinh \mu, 0, 0), \quad (20)$$

parametrized by a rapidity  $\mu$  satisfying the normalization  $u^c u_c = -1$ . Here,  $u^r < 0$  ( $\mu > 0$ ) refers to radial infall. Here,  $u^t = dt/d\tau$  combines the combined result of gravitational redshift and Lorentz time-dilation, associated with the Lorentz factor along the trajectory.

In between successive photon emission events,  $M$  is constant. Over the course of a time interval  $N_\gamma^{-1}$ , the redshift factor reduces according to

$$\alpha^2 = e^{-\lambda} \sim e^{-\frac{\Delta t}{2(M+E)}} \sim e^{-2650}, \quad (21)$$

i.e., zero in any quantization scheme of particle trajectories. Between any two photon emissions, therefore,

$\mathcal{O}$  has completely settled down to  $r \simeq 2(M + E)$  as viewed from infinity. This asymptotic behavior hereby represents a singular limit for  $u^t$ . Absent photon emissions,  $(\partial_t)^b = (1, 0, 0, 0)$  is a Killing vector, and the four-momentum  $p^b = m u^b$  satisfies conservation of total energy-at-infinity

$$E = m\eta = m\alpha^2 u^t = m\alpha \cosh \mu. \quad (22)$$

The normalization condition  $u^c u_c = -1$  gives  $\dot{r}^2 = \eta^2 - \alpha^2 \simeq \eta^2$  in the saturation limit when  $\alpha^2$  is close to zero, whereby

$$2\dot{M} \simeq \dot{r} \simeq -\eta. \quad (23)$$

With  $dM/d\tau = u^t dM/dt = -Lu^t$ , it implies a finite  $u^t = dt/d\tau$ , i.e.,

$$\frac{d\tau}{dt} \simeq \frac{2L}{\eta}. \quad (24)$$

The limit  $L = 0$  recovers the conventional limit  $d\tau/dt = 0$  for a test particle falling onto a classical Schwarzschild black hole.

According to the geodesic equations of motion  $u^c \nabla_c u^b = 0$ ,  $du^b/d\tau + \Gamma_{ac}^b u^a u^c = 0$ . Between photon emissions  $t_i < t < t_{i+1}$ , we retain Christoffel symbols containing the impulsive changes  $\delta M$ , given by

$$\Gamma_{tt}^t = \frac{\dot{M}}{r - 2M}, \Gamma_{rr}^t = \frac{\dot{M} r^2}{(r - 2M)^3}, \Gamma_{rt}^r = \frac{\dot{M}}{r - 2M}. \quad (25)$$

With  $du^b/d\tau = u^t (du^b/dt)$ , we obtain

$$(u^t)^{-1} \frac{du^t}{dt} + \Gamma_{tt}^t + \Gamma_{rr}^t \left( \frac{u^r}{u^t} \right)^2 \simeq 0, \quad (26)$$

$$(u^r)^{-1} \frac{du^r}{dt} + 2\Gamma_{rt}^r \simeq 0. \quad (27)$$

With  $\dot{M} \delta t = -\epsilon$  and

$$\frac{\delta u^t}{u^t} \simeq \frac{\delta \eta}{\eta} + e^{-\lambda} \delta e^\lambda, \quad \frac{\delta u^r}{u^r} \simeq \frac{\delta \eta}{\eta} + \frac{2}{\sinh(2\mu)} \delta \mu, \quad (28)$$

we arrive at  $\delta \eta/\eta + e^{-\lambda} \delta e^\lambda \simeq (\epsilon/r\alpha^2) (1 + \tanh^2 \mu)$ , i.e.,

$$\frac{\delta \eta}{\eta} + e^{-\lambda} \delta e^\lambda \simeq \frac{\epsilon}{M} \left( \frac{1 + \tanh^2 \mu}{2} \right) e^\lambda. \quad (29)$$

Similarly,  $\delta \eta/\eta + 2\delta \mu/\sinh(2\mu) \simeq (2\epsilon/r)$  gives

$$\frac{\delta \eta}{\eta} + \frac{2\delta \mu}{\sinh(2\mu)} \simeq \frac{\epsilon}{M}. \quad (30)$$

In the ultra-relativistic limit (large  $\mu$ ), (29) and (30) reduce to

$$\frac{\delta \eta}{\eta} + e^{-\lambda} \delta e^\lambda \simeq \frac{\epsilon}{M} e^\lambda, \quad \frac{\delta \eta}{\eta} \simeq \frac{\epsilon}{M}. \quad (31)$$

#### IV. EM IS CONSTANT

The ultra-relativistic limit (31) relates  $(\delta\eta, \delta M)$  across instances of photon emission. With  $\epsilon = -\delta M$  and constant  $m$ , (31) gives the integral of motion

$$EM \simeq mj \quad (32)$$

for some constant  $j$ . By (23-24), we have

$$3kt_{ev} = M_0^3, \quad (33)$$

where  $k$  is short-hand in the black hole luminosity  $dM/dt = -kM^{-2}$ . Hence, (32) gives  $j = (2k/M)u^t$ . By (23) and (32), furthermore,

$$M(\tau) \simeq M_0 \left(1 - \frac{j\tau}{M_0^2}\right)^{\frac{1}{2}}, \quad (34)$$

establishing the evolution of  $M$  observed by  $\mathcal{O}$ .

The finite value (24) of the correlation  $u^t$  between  $t$  and  $\mathcal{O}$ 's eigentime can be further expressed in terms of a finite distance to the event horizon, as seen in the Schwarzschild line-element of a ZAMO. By differentiation of (34),

$$\frac{t}{t_{ev}} \simeq 1 - \left(1 - \frac{j\tau}{M_0^2}\right)^{\frac{3}{2}}. \quad (35)$$

we have

$$u^t = \frac{3}{2} \left(\frac{t_{ev}}{M_0}\right) \left(\frac{M}{M_0}\right) \frac{j}{M_0}. \quad (36)$$

According to (22), we have  $u^t \simeq (E/m)e^\lambda$ , so that

$$e^\lambda \simeq \left(\frac{m}{E}\right) u^t = \frac{3}{2} \left(\frac{t_{ev}}{M_0}\right) \left(\frac{M}{M_0}\right)^2. \quad (37)$$

That is

$$e^\lambda \simeq \frac{M^2}{2k} = -\frac{1}{2} \left(\frac{dM}{dt}\right)^{-1} = 960C \left(\frac{M}{\epsilon}\right), \quad (38)$$

where we used (13) and (33). It follows that (38) implies

$$\delta\lambda = \ln(960C + 1) \simeq 8.58 \quad (39)$$

at each photon emission. During the (on average) time interval  $\Delta t = 1920CM$  between photon emissions, this shift (39) relaxes again to (38). This result is insensitive to  $\delta U$ .

#### V. CONCLUSIONS

Hawking radiation is identified with discrete evolution of a Schwarzschild space time, operating essentially at the quantum computation limit allowed by its total mass-energy at infinity. In particular, each photon emission results from an astronomically large calculation that, accumulated over all photon emissions, results in a total calculation quantitatively in agreement with a unitary  $S$ -matrix formulation of complete black hole evaporation.

The discrete evolution (3) of space-time with long time intervals  $t_i < t < t_{i+1}$  of static space-time extending over a few thousand light crossing time scales enables us to calculate in detail the trajectory of an observer in free fall by integration of the geodesic equations of motion. The result shows a correlation (32) between  $\mathcal{O}$ 's total mass energy  $E$  and  $M$ , which reduces to (32).

In a holographic interpretation [15], (4) is the information on screens enveloping the event horizon, encoding the position of a central singularity of mass  $M$  at a radial distance  $R_S = 2R_g$ . (The appearance of the event horizon is finite temperature effect by acceleration of the screen, positioned at the constant Schwarzschild radius  $R_S$ . At the zero temperature of a screen in free fall, such horizon is absent.) A mass-energy  $E \ll M$  falling in adds to  $S$  an additional information  $8\pi EM$ . By (32), it is conserved during black hole evaporation, as long as the accretion condition  $E > k_B T$  is satisfied.

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